Fourier Series: Forms & Equations

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$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t}$$

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n)$$

Complex exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} \,, \qquad \omega_0 = \frac{2\pi}{T_0}$$

Trigonometric Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \,, \quad \omega_0 = \frac{2\pi}{T_0}$$

Compact Fourier series

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \varphi_n) \,, \qquad \omega_0 = \frac{2\pi}{T_0}$$

Compact to Trigonometric form

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

Use the trigonometric identity $A\cos(\beta) + B\sin(\beta) = C\cos(\beta - \theta)$ to get,

$$c_n \cos(n\omega_0 t - \theta_n) = a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

where

$$c_n = \sqrt{a_n^2 + b_n^2}\ , \qquad n = 0, 1, 2, 3, \ldots$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)\ , \qquad n = 0, 1, 2, 3, \ldots$$

Notice that b_0 does not exist, i.e., $c_0 = \sqrt{a_0^2 + 0} = a_0$ (see above).

Trigonometric to Compact form

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n)$$

Use the inverse trigonometric identifies,

$$a_n = c_n \cos(\theta_n)\,, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = c_n \sin(\theta_n)\,, \qquad n = 0, 1, 2, 3, \ldots$$

Notice that θ_0 does not exist, i.e., $a_0 = c_0 \cos(0) = c_0$ (see above).

Easy to memorize: complex numbers, 2D vectors and trigonometric identities all have similar looking equations (*different* topics but *similar* equations).

Compact to Complex Exponential form

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t}$$

Use Euler's identity $e^{j\beta} = \cos(\beta) + j\sin(\beta)$ to get

$$\cos(\beta) = \frac{e^{j\beta} + e^{-j\beta}}{2}$$

$$\sin(\beta) = \frac{e^{j\beta} - e^{-j\beta}}{2j}$$

Compact to Complex Exponential form

$$\begin{aligned} x(t) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \\ &= \frac{c_0}{2} + \sum_{n=1}^{\infty} \frac{\left[c_n e^{j(n\omega_0 t - \theta_n)} + c_n e^{-j(n\omega_0 t - \theta_n)}\right]}{2} \\ &= \frac{c_0}{2} + \sum_{n=1}^{\infty} \frac{\left[c_n e^{-j\theta_n}\right]}{2} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{\left[c_n e^{j\theta_n}\right]}{2} e^{-jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} \end{aligned}$$

where

$$\alpha_n = \frac{c_n}{2} e^{-j\theta_n} \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$|\alpha_n| = \frac{c_n}{2} \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$\angle\alpha_n = -\theta_n \ , \qquad n = 0, 1, 2, 3, \ldots$$

and for real-valued $x(t)$

$$\alpha_{-n} = \alpha_n^* \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$|\alpha_{-n}| = |\alpha_n| = \frac{c_n}{2} \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$\angle\alpha_{-n} = -\angle\alpha_n = \theta_n \ , \qquad n = 0, 1, 2, 3, \ldots$$

Notice that the extra DC part is

$$\alpha_0 = \frac{c_0}{2} = \frac{a_0}{2}$$

Complex Exponential to Compact form

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n)$$

$$c_n = 2|\alpha_n| \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$\theta_n = -\angle\alpha_n = \tan^{-1}\left(\frac{-\mathrm{Im}\{\alpha_n\}}{\mathrm{Re}\{\alpha_n\}}\right) , \qquad n = 0, 1, 2, 3, \ldots$$

Complex Exponential to/from Trigonometric form

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$|\alpha_n| = \frac{c_n}{2} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \ , \ \angle\alpha_n = -\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \ , \quad n = 0, 1, 2, 3, \ldots$$

$$\alpha_n = \frac{a_n}{2} - j\frac{b_n}{2} \ , \qquad n = 0, 1, 2, 3, \ldots$$

Also,

$$a_n = 2 \, \mathrm{Re}\{\alpha_n\} \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = -2 \, \mathrm{Im}\{\alpha_n\} \, , \qquad n = 0, 1, 2, 3, \ldots$$

The values a_n, b_n, c_n, θ_n, $\alpha_n = |\alpha_n|\angle\alpha_n$ are known as the **Fourier series coefficients**. You need to find them for each periodic signal $x(t)$ you want to analyze. The frequencies are always $\omega_n = n\omega_0 = n(2\pi/T_0)$.

Complex Exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t}$$

$$\alpha_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt \quad , \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

For real-valued signal $x(t)$

$$\alpha_{-n} = \alpha_n^* \ , \qquad n = 0, 1, 2, 3, \ldots$$

Trigonometric Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t)\, dt \quad , \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(n\omega_0 t)\, dt \quad , \qquad n = 0, 1, 2, 3, \ldots$$

If $x(t)$ is even, then $b_n = 0, \forall n$, and $a_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \cos(n\omega_0 t)\, dt$.

If $x(t)$ is odd, then $a_n = 0, \forall n$, and $b_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \sin(n\omega_0 t)\, dt$.

Compact Fourier series

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n)$$

$$c_n = \sqrt{a_n^2 + b_n^2} \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$c_n = 2|\alpha_n| \ , \qquad n = 0, 1, 2, 3, \ldots$$

$$\theta_n = -\angle \alpha_n = \tan^{-1}\left(\frac{-\mathrm{Im}\{\alpha_n\}}{\mathrm{Re}\{\alpha_n\}}\right) , \qquad n = 0, 1, 2, 3, \ldots$$

Notice that $\alpha_0 = \frac{c_0}{2} = \frac{a_0}{2}$ is the **average value** or **DC value** or **offset** or **vertical shift** of the signal

$$\alpha_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{0}dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)dt$$

$$\frac{a_0}{2} = \frac{1}{2}\times\frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(n\omega_0 t)\,dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(0)\,dt$$
$$= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)dt$$

For some signals, the DC value can be found simply by inspection.

It is interesting to note that sinusoids with frequencies that are multiple of the fundamental frequency are **orthogonal signals**.

The same thing can be said about exponentials with frequencies that are multiple of the fundamental frequency. They are **orthogonal signals**.

In general, two signals $x(t)$ and $y(t)$ are orthogonal over the interval (t_0, t_1) if they satisfy the condition

$$\int_{t_0}^{t_1} x(t)\, y(t)^*\, dt = 0$$

Other popular orthogonal signals include DC and AC, and also $\cos(n\omega_0 t)$ and $\sin(n\omega_0 t)$ of the exact same frequency (due to the 90° phase shift).

Convergence of Fourier series: Periodic signal $x(t)$ has a Fourier series representation if it satisfies the following Dirichlet conditions:

1. $x(t)$ is absolutely integrable over any period, that is,

$$\int_{t_0}^{t_0+T_0} |x(t)|\, dt < \infty$$

2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.

3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

Practical signals satisfy these conditions. These conditions are sufficient but not necessary conditions for the Fourier series representation.